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Nonlinear oscillations of non-spherical cavitation bubbles in acoustic fields

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The nonlinear stability of gas bubbles in acoustic fields is studied using a multiple-scale type of expansion. In particular the development of a subharmonic or a synchronous perturbation to the flow is investigated. It is shown when an equilibrium non-spherical shape oscillation of a bubble is stable. If the amplitude of the sound field is ϵ then it is shown that subharmonic perturbations of order $\epsilon^{\frac{1}{2}}$ can exist and be stable. Furthermore synchronous perturbations of order ϵ can exist and be stable. It is shown that synchronous perturbations, unlike the subharmonic case where the bifurcation is symmetric, bifurcate transcritically when the driving frequency is varied and also undergo secondary bifurcations. It is further shown that, in certain cases, the latter properties of the synchronous modes cause the flow to exhibit a hysteresis phenomenon when the driving frequency is varied.

1. Introduction

Radial oscillations of cavitation bubbles in acoustic fields have been the subject of many investigations both in linear and in nonlinear regimes (see Plesset & Prosperetti 1977 for a recent review of the literature). Much of this work originated as an attempt to explain the experimentally observed connexion between the appearance of a signal at half the frequency of the acoustic wave and the onset of cavitation developing in an acoustically irradiated liquid. The matter has not been wholly settled yet. However it has been shown that the instability of the purely harmonic motion can definitely lead to a subharmonic response, depending on the values of the relevant parameters and on the initial conditions.

Non-spherical oscillations of cavitation bubbles are less well understood. It is known (Elder 1959; Gould 1966) that experimentally these surface oscillations arise as a bifurcation from the basic purely radial oscillatory motion. Such bifurcations occur in relatively low-viscosity liquids as the amplitude of the acoustic field exceeds a threshold dependent on the relevant parameters of the problem. Further increase of the wave amplitude leads to further bifurcations and eventually to a chaotic surface agitation.

In addition to the onset of surface oscillations the presence of small-scale streaming was detected in the above experiments. This effect, often called 'microstreaming', has never been properly understood. Davidson & Riley (1971) proposed an explanation in terms of the steady streaming induced by a rigid sphere oscillating along a diameter. However the latter phenomenon occurs continuously however small the amplitude of the oscillation, whereas the onset of 'microstreaming' is associated with a bifurcation, at least for a range of sonic amplitudes.

The importance of microstreaming as a mechanism for the acceleration of rate processes (rectified diffusion and heat transfer) has been discussed by Eller (1969) and Gould (1974) among others.

Another phenomenon displayed by bubbles caused to pulsate by a sound field was first detected by Benjamin & Strasberg (1958) and subsequently observed by Eller & Crum (1970). A bubble, trapped and held in a fixed position by an acoustic standing wave, begins to move erratically if a threshold wave amplitude is exceeded. The suggestion originally made by Strasberg & Benjamin (1958), that the dancing motion might be caused by the presence of surface oscillations, was then rejected by the same authors owing to the inadequate agreement between theoretical and experimental results. The work of Eller & Crum (1970) concludes that a calculation of the threshold amplitude for surface oscillations more accurate than those performed by Benjamin & Strasberg (1958) and Hsieh & Plesset (1961) leads to a better agreement with the experimental threshold for the erratic dancing motion of bubbles. Thus Eller & Crum (1970) support the original suggestion of Benjamin & Strasberg (1958).

The aim of the present work is to follow the perturbation, which the linear theory predicts to grow fastest for suitable values of the parameters of the problem, into the nonlinear regime. In the absence of any damping effect, and assuming that the perturbations are axisymmetric, the basic radial oscillatory motion will be shown to bifurcate into a non-spherical flow which is periodic in the fast time scale imposed by the acoustic excitation and is slowly modulated on the time scale associated with the linear growth rate. The perturbations are found to eventually reach equilibrium amplitudes given by the singular points of the amplitude equations derived in § 5. We also show that a steady-streaming component may be present in the bifurcated flow under suitable conditions. Furthermore we discuss how the present theory can explain the mechanism whereby surface oscillations can excite the displacement of the centre of gravity of the bubble, thus leading to the dancing mentioned above.

The analysis will be restricted to small amplitudes of the acoustic wave. Any fundamental or ultraharmonic resonance (see Prosperetti 1974) will also be assumed absent in the basic radial motion. Under the above conditions the problem will be shown to be amenable to analytical treatment.

The procedure adopted in the rest of the paper is the following. In the next section we formulate the problem in non-dimensional form. In § 3 we derive a basic radial solution of the fundamental equations, under non-resonant conditions, as a power series of the 'small' amplitude of the acoustic wave. Section 4 is devoted to a linear stability analysis; and particularly to the cases of subharmonic and synchronous responses of the perturbations. The components of the perturbations which linear theory predicts to grow fastest are followed in a weakly nonlinear regime in § 5 both for the subharmonic and the synchronous cases. The analysis leads to amplitude equations whose solutions are discussed in § 6. A discussion of the implications of the present results follows in § 7.

2. Formulation of the problem

We consider a single cavity, filled with a permanent non-condensable gas, immersed in an unbounded liquid. We neglect any effect of the vapour necessarily present in the cavity.

The bubble is set into pulsation by a sound field of wavelength large compared with the bubble radius. Thus the presence of the acoustic wave can be modelled in an approximate way by assuming that far enough from the bubble the pressure is given by

$$p^* = P^*_\infty(1+\epsilon\cos\omega t^*), \tag{1}$$

where P^*_∞ is the average ambient pressure, t^* is time, ω is the frequency and ϵ is the amplitude of the wave. (The asterisk denotes dimensional quantities.) Furthermore the flow in the liquid can be assumed to be incompressible.

We restrict our attention to small-amplitude acoustic waves, and thus write

$$\epsilon \ll 1. \tag{2}$$

The pulsations of the cavity, forced by the sound field, will be assumed to occur around the equilibrium radius R_0 defined by

$$R_0(P^*_0 - P^*_\infty) = 2\sigma \tag{3}$$

where σ is surface tension and P^*_0 is the equilibrium internal pressure of the cavity. The effect of any mechanism which can alter the value of R_0, like rectified diffusion, will be neglected. Indeed such a mechanism would lead to a parametric variation of R_0 on a time scale much slower than those relevant for the instability problem treated here.

Besides neglecting the acoustic radiation of energy, we will neglect viscous effects in the liquid. Since the influence of viscosity is presumably confined within a layer of typical thickness $(2\nu/\omega)^{\frac{1}{2}}$ the inviscid assumption will be justified for relatively high frequency oscillations and relatively large bubble radii.

Finally we neglect the motion of the gas inside the cavity and assume that its pressure p^*_i is uniformly distributed according to the isothermal law

$$p^*_i/P^*_0 = V^*_0/V^* \tag{4}$$

where V^*_0 is the volume of the cavity at equilibrium, V^* is its volume at time t^*.

The assumption that the pressure is uniform in the cavity is justified if the size of the cavity is small compared to the acoustic wavelength in the gas. Furthermore we are justified in assuming that the gas behaves isothermally if the thermal penetration depth in the liquid is small compared to the size of the cavity. If the latter ratio is large then the gas behaves adiabatically. We do not pursue this case but it can be shown that the qualitative description of the bifurcating solutions in this case is similar to that appropriate to the isothermal case. A more detailed discussion of the assumptions leading to (4) is given by Plesset & Prosperetti (1977).

Let us then consider a general non-spherical motion of the cavity characterized by the following equation of the interface.

$$F(r^*, \theta, \phi, t^*) = 0 \tag{5}$$

with (r^*, θ, ϕ) spherical polar co-ordinates.

Under the above conditions the differential problem governing the inviscid irrotational flow of the liquid can be written

$$\nabla^2\Phi = 0, \tag{6}$$

$$\left\{\frac{\partial F}{\partial t} - (\nabla F).(\nabla\Phi)\right\}_{F=0} = 0, \tag{7}$$

$$\left\{p_i - (1+\epsilon\cos\Omega t) - \mathscr{S}\left(\frac{1}{\mathscr{R}_1}+\frac{1}{\mathscr{R}_2}\right) - \frac{\partial\phi}{\partial t} + \tfrac{1}{2}(\nabla\Phi)^2\right\}_{F=0} = 0, \tag{8}$$

where the variables have been made non-dimensional in the form

$$t = t^*\left(\frac{P^*_\infty}{\rho R_0^2}\right)^{\frac{1}{2}}, \quad r^* = rR_0, \quad p^* = P^*_\infty p, \quad \phi^* = R_0\left(\frac{P^*_\infty}{\rho}\right)^{\frac{1}{2}}\Phi$$

and $(\mathscr{R}_1, \mathscr{R}_2)$ are the non-dimensional principal radii of curvature of the interface. The surface tension parameter $\mathscr{S}$ and the frequency parameter Ω are defined by

$$\mathscr{S} = \frac{\sigma}{P^*_\infty R_0}, \quad \Omega = \omega R_0\left(\frac{\rho}{P^*_\infty}\right)^{\frac{1}{2}}, \tag{9}$$

and the non-dimensional velocity vector $\mathbf{v}$ is defined as $(-\nabla\phi)$. Finally the internal pressure of the cavity p_i has the non-dimensional form:

$$p_i = \frac{(1+2\mathscr{S})}{V(t)} \tag{10}$$

where $V(t)$ is the non-dimensional volume of the cavity at time t. Further we note that (7) and (8) are the kinematic and dynamic boundary conditions respectively to be satisfied on the boundary of the bubble.

3. Basic flow

The governing differential problem (6), (7), (8) admits a purely radial solution such that

$$F = r - R(t) = 0, \tag{11}$$

$$\Phi = \frac{R^2\dot{R}}{r}, \tag{12}$$

with $R(t)$ the solution of the well-known generalized Rayleigh equation

$$R\ddot{R} + \tfrac{3}{2}\dot{R}^2 + (1+\epsilon\cos\Omega t) + \frac{2\mathscr{S}}{R} - \frac{1+2\mathscr{S}}{R^3} = 0. \tag{13}$$

Periodic solutions of (13) can be easily obtained as power series in the parameter ϵ, provided the frequency Ω does not attain a value close to an integral fraction or multiple of the natural frequency Ω_0 defined by

$$\Omega_0^2 = 3 + 4\mathscr{S}. \tag{14}$$

Using such a procedure we find

$$R = 1 + \epsilon(R_1\cos\Omega t) + \epsilon^2\{R_{20} + R_{22}\cos 2\Omega t\} + O(\epsilon^3), \tag{15}$$

where

$$R_1 = (\Omega^2-\Omega_0^2)^{-1} \tag{16}$$

$$R_{20} = \Omega_0^{-2}(5\Omega_0^2-\Omega^2-3)\,R_1^2, \tag{17}$$

$$R_{22} = (4\Omega_0^2-16\Omega^2)^{-1}\{5(\Omega_0^2+\Omega^2)-3\}\,R_1^2. \tag{18}$$

4. Linear stability

We consider a perturbed interface such that

$$F = r-R(t)-g(\theta,\phi,t), \tag{19}$$

and assume the perturbation g to be infinitesimal. Moreover we expand g in terms of a complete set of orthonormal functions and write

$$g(0,\phi,t) = \sum_{n=0}^{\infty}\sum_{m=-n}^{\infty} G_n^m(t)\,Y_n^m(\theta,\phi), \tag{20}$$

with Y_n^m the spherical harmonic of order (n, m) and G_n^m a function of time to be determined. The corresponding form of the perturbed velocity potential Φ can be written in terms of unknown functions $\Phi_n^m(t)$ in the form:

$$\Phi = \frac{R^2\dot{R}}{r}+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Phi_n^m\,Y_n^m r^{-(n+1)}. \tag{21}$$

On substituting (20) and (21) into (7), (8) and neglecting nonlinear terms we find that, as found for example by Plesset & Mitchell (1950), y satisfies

$$\ddot{y}+\left\{\frac{\mathscr{S}(n+2)(n^2-1)}{R^3}-\frac{3}{4}\frac{\dot{R}^2}{R^2}-\left(n+\frac{1}{2}\right)\frac{\ddot{R}}{R}\right\}y = 0, \tag{22}$$

where

$$y = G_n^m R^{\frac{3}{2}}. \tag{23}$$

In the general case of large amplitude forcing, the equation (22) is of the Hill type and must be solved numerically. In the present small-amplitude case, after defining

$$T = \tfrac{1}{2}\Omega t, \tag{24}$$

and using (15), the equation takes the form

$$\frac{d^2y}{dT^2}+\{a+b\cos 2T+(c+d\cos 4T)\,\epsilon^2+O(\epsilon^3)\}\,y = 0, \tag{25}$$

with

$$a = (\Omega_0^2-3)\,\Omega^{-2}(n+2)(n^2-1) \tag{26a}$$

$$b = 4R_1\{(n+\tfrac{1}{2})-\tfrac{3}{4}\Omega^{-2}(\Omega_0^2-3)(n+2)(n^2-1)\}, \tag{26b}$$

$$c = \{3a-\tfrac{3}{2}-(2n+1)\}\,R_1^2-3aR_{20}, \tag{26c}$$

$$d = \{3a+\tfrac{3}{2}-(2n+1)\}\,R_1^2+\{8(2n+1)-3a\}\,R_{22}. \tag{26d}$$

The equation (25) is still of the Hill type and coincides with that derived by Eller & Crum (1970). The equations discussed by Benjamin & Strasberg (1958) and Hsieh

& Plesset (1961) can be obtained from (25) if terms $O(\epsilon^2)$ are neglected. Such equations are of the Mathieu type. Thus the solutions of (25) can be put in the form

$$y(T) = \exp(\mu T)\, P(T) \tag{27}$$

with $P(T)$ a periodic function of time. Furthermore it is well known that unstable ($\mathrm{Re}(\mu) > 0$) regions exist in the (a, ϵ) plane close to the points $(N^2, 0)$ with $N = 1, 2, 3, 4, \ldots$.

We consider separately the behaviour of the linear solution in a neighbourhood of the points $(1, 0)$ and $(4, 0)$.

(a) Subharmonic case

Let us show that in a neighbourhood of $(1, 0)$ the response of the disturbance to the basic oscillation is subharmonic and $\mu \sim O(\epsilon)$.

Let us set up the following expansions

$$y = y_0 + \epsilon y_1 + O(\epsilon^2), \tag{28}$$

$$a = 1 - 2\lambda\epsilon + O(\epsilon^2), \tag{29}$$

$$\Omega = \tilde{\Omega}(1 + \lambda\epsilon) + O(\epsilon^2), \tag{30}$$

where

$$\tilde{\Omega} = \{(\Omega_0^2 - 3)(n+2)(n^2-1)\}^{\frac{1}{2}}. \tag{31}$$

The time dependence of the perturbations is described by two time variables, a fast one T associated with the forced oscillation, and a slow variable τ defined by

$$\tau = \epsilon T. \tag{32}$$

If the above expansions are substituted into (25) and the differential systems obtained by equating terms of order ϵ^0 and ϵ are solved we can show that, correct to order ϵ,

$$\mu = \pm(\lambda^2 - \tfrac{1}{4}b^2)^{\frac{1}{2}}\epsilon, \tag{33}$$

so that the flow is unstable if $\lambda^2 > \frac{1}{4}b^2$.

(b) Synchronous case

We now consider the case $a = 4$ and show that the linear response of the perturbation is synchronous with the forcing and $\mu \simeq O(\epsilon^2)$. Let us set up the expansions

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + O(\epsilon^3), \tag{34}$$

$$a = 4(1 - 2\lambda\epsilon^2) + O(\epsilon^4), \tag{35}$$

$$\Omega = \frac{\tilde{\Omega}}{2}(1 + \lambda\epsilon^2) + O(\epsilon^2), \tag{36}$$

and now take $\tau = \epsilon^2 T$. If we substitute the above expansions into (25) and solve the resulting differential equations obtained by equating like powers of ϵ we can show that, correct to order ϵ^2, the growth rate μ is now given by

$$\mu = \pm\frac{1}{4}\left\{\left(\frac{b_0^2}{48} + c_0 - \frac{d_0}{2} - 8\lambda\right)\left(\frac{5}{48}b_0^2 - c_0 - \frac{d_0}{2} + 8\lambda\right)\right\}^{\frac{1}{2}}\epsilon^2, \tag{37}$$

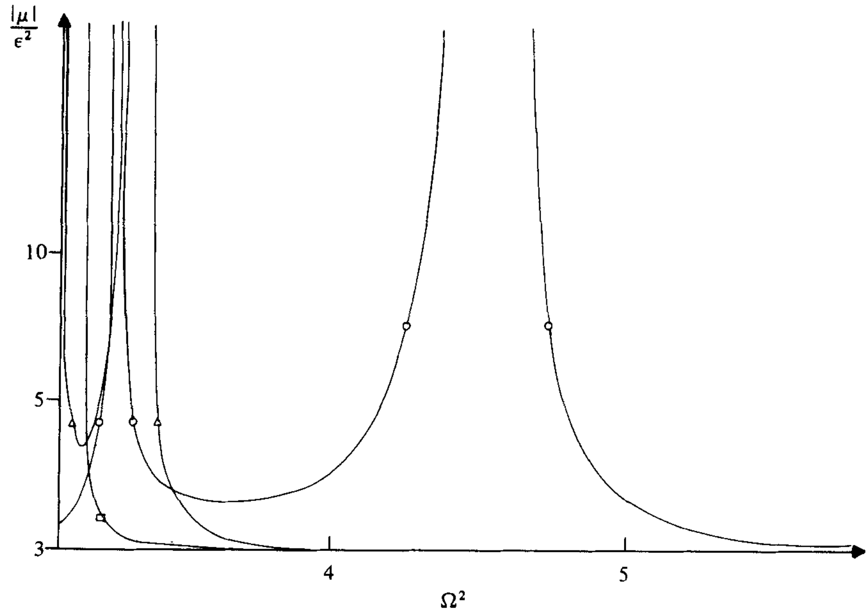

FIGURE 1. The growth rates of the harmonics $n = 2, 3, 4$ plotted versus Ω^2 for the synchronous case. θ, $n = 2$; $\triangle$, $n = 3$; $\boxminus$, $n = 4$.

where c_0, d_0, b_0 are given by (26*b*), (26*c*), (26*d*) respectively with Ω replaced by $\frac{1}{2}\tilde{\Omega}$. Thus the flow is unstable for values of λ such that

$$\left(\frac{5}{48}b_0^2 - c_0 - \frac{d_0}{2}\right) < 8\lambda < \left(\frac{b_0^2}{48} + c_0 - \frac{d_0}{2}\right). \tag{38}$$

From (37) one can compute the growth rate of the perturbations as a function of Ω_0 and n within the unstable region given by (38). We can use the above results to determine approximations to the growth rate for the subharmonic and synchronous cases. However it is clear that in both cases the growth rates depend on n. In order to make any analytical progress we restrict our attention to the case $n = 2$. In some cases (see, for example, Hsieh 1974) other modes can be more unstable but, in general, the $n = 2$ mode is the most dangerous. In order to illustrate this point we have in figure 1 some plotted results for the synchronous case which enable us to predict that, within the inviscid scheme the fastest growing component of the perturbation is that corresponding to $n = 2$ except for a relatively small range of values of Ω_0^2 close to 3 where resonances of the higher harmonics are excited. Indeed the harmonic N resonates when $(\Omega)_{n=N}$ equals $(4)^{1-\frac{1}{2}m}\,\Omega_0$ with $m = 1, 2, 3, \ldots$ and one can readily see that the resonance regions cluster in a neighbourhood of the origin of figure 1. We also notice that the values of $|\mu|/\epsilon^2$ plotted in figure 1 are the maximum values in the unstable range (38).

5. Nonlinear stability

We now follow in the nonlinear regime the evolution of the component $n = 2$ of the perturbation. We restrict our attention to ϕ-independent disturbances ($m = 0$). This assumption, while simplifying the analysis considerably, does not seem to alter the essential features of the nonlinear mechanism involved. We consider the subharmonic case and the synchronous case separately.

(*a*) *Subharmonic case*

Let us take a and Ω to be given by (29), (30) respectively, and consider the weakly nonlinear process arising when the fundamental ($n = 2$) interacts with itself and the basic flow.

The first-order interaction between the fundamental and itself reproduces the fundamental and gives rise to harmonics of order 4 and 0 (the latter represents a distortion of the basic flow). This behaviour depends on the properties of integrals of products of three spherical harmonics. The above components are both functions of the slow time variable τ (as defined by (32)) and of the fast variable T (as defined by (24)) through second-order products of e^{iT}, e^{-iT}. Second-order interactions between the harmonics generated at first-order and the fundamental reproduce the fundamental besides generating further lower and higher harmonics. The fast time dependence of the fundamental is also reproduced so that an orthogonality condition is required at this order to avoid the occurrence of secular terms.

The latter requirement determines the order of magnitude of the fundamental which is found to be $\epsilon^{\frac{1}{2}}$. The interaction of the basic flow with the fundamental leads to an $O(\epsilon^{\frac{3}{2}})$ contribution.

The above arguments suggest that the following expansions be considered

$$r = 1+\epsilon^{\frac{1}{2}}f_1+\epsilon f_2+\epsilon^{\frac{3}{2}}f_3+O(\epsilon^2), \tag{39}$$

$$\Phi = R^2\dot{R}r^{-1}+\epsilon^{\frac{1}{2}}\phi_1 r^{-3}+\epsilon(\phi_0 r^{-1}+\phi_2 r^{-3}+\phi_4 r^{-5})+\epsilon^{\frac{3}{2}}\phi_3 r^{-3}+O(\epsilon^2), \tag{40}$$

where
$$f_1 = g_1, \quad f_2 = \rho_1+g_0+g_2+g_4, \quad f_3 = g_3, \qquad (41a, b, c)$$

$$(g_1,\phi_1) = (G_{21}(\tau,T),\phi_{21}(\tau,T))P_2, \quad (g_2,\phi_2) = (G_{22}(\tau,T),\phi_{22}(\tau,T))P_2,$$
$$(g_0,\phi_0) = (G_{01}(\tau,T),\phi_{01}(\tau,T)), \quad \rho_1 = R_1\cos 2T,$$
$$(g_4,\phi_4) = (G_{41}(\tau,T),\phi_{41}(\tau,T))P_4, \quad (g_3,\phi_3) = (G_{23}(\tau,T),\phi_{23}(\tau,T))P_2$$
$$+\text{terms orthogonal to } P_2. \qquad (42a\text{–}f)$$

The bracket notation should not be confused with the scalar product introduced later in this section.

Furthermore R_1 is given by (16) with Ω replaced by $\tilde{\Omega}$ and $P_N(\cos\theta)$ is the associated Legendre polynomial of order N. After substituting (40), (41), (42) into (7) and (8) and defining

$$' \equiv \frac{\tilde{\Omega}}{2}\frac{\partial}{\partial T} \tag{43}$$

much algebraic work leads to the following form of the kinematic and dynamic boundary conditions

$$\left\{\frac{\partial F}{\partial t}-(\nabla F).(V\phi)\right\}_{F=0} = \epsilon^{\frac{1}{2}}\{3\phi_1-g_1'\}+\epsilon\{-g_0'-g_2'-g_4'-12g_1\phi_1+\phi_0+3\phi_2+5\phi_4+g_{1\theta}\phi_{1\theta}\}$$
$$+\epsilon^{\frac{3}{2}}\Big\{-g_3'^{3}\phi_3-2g_1\rho_1'-12\rho_1\phi_1+30\phi_1 g_1^2-12g_0\phi_1$$
$$-12g_2\phi_1-12g_4\phi_1-2g_1\phi_0+12g_1\phi_2-30\phi_4 g_1-5g_1g_{1\theta}\phi_1+g_{1\theta}\phi_{2\theta}$$
$$+g_{1\theta}\phi_{4\theta}+g_{2\theta}\phi_{1\theta}+g_{4\theta}\phi_{1\theta}-\frac{\tilde{\Omega}}{2}\frac{\partial g_1}{\partial\tau}+\lambda g_1'\Big\}+O(\epsilon^2) = 0, \tag{44}$$

$$
\begin{aligned}
&\epsilon^{\frac{1}{2}}\{-\phi_1'-(\Omega_0^2-3)\,g_1\}+\epsilon\{\tfrac{9}{2}\phi_1^2+\tfrac{1}{2}\phi_{1\theta}^2+3g_1\phi_1'-\phi_0'-\phi_2'-\phi_4' \\
&\quad+(\Omega_0^2-3)\,(-g_2-\tfrac{9}{2}g_4+\tfrac{5}{2}g_1^2)-\Omega_0^2G_{01}-\tfrac{3}{10}(\Omega_0^2-1)\,(G_{21})^2-\Omega_0^2\rho_1-\rho_1''\} \\
&\quad+\epsilon^{\frac{3}{2}}\Big\{-\phi_3'-(\Omega_0^2-3)\,g_3+3\rho_1\phi_1'+\rho_1''g_1+3\phi_1\rho_1'+2\rho_1(\Omega_0^2-3)\,g_1 \\
&\quad+3g_0\phi_1'+3g_2\phi_1'+3g_4\phi_1'+g_1\phi_0'+3g_1\phi_2'+5g_1\phi_4' \\
&\quad+3\phi_1\phi_0+9\phi_1\phi_2+15\phi_1\phi_4+\phi/_{1\theta}\phi_{2\theta}+\phi_{1\theta}\phi_{4\theta} \\
&\quad-4g_1\phi_{1\theta}^2-36g_1\phi_1^2-6g_1^2\phi_1'+(\Omega_0^2-3) \\
&\quad\times(-4g_1^3+2g_1g_0+5g_1g_2+12g_1g_4+\tfrac{3}{4}g_1g_{1\theta}^2-\tfrac{1}{4}g_{1\theta\theta}g_{1\theta}^2) \\
&\qquad-\tfrac{1}{2}(\Omega_0^2-1)\,(\tfrac{6}{5}G_{21}G_{22}+\tfrac{2}{3}G_{21}G_{41}+\tfrac{2}{35}(G_{21})^2)+\frac{\tilde{\Omega}}{2}\frac{\partial\phi_1}{\partial\tau}-\lambda\phi_1'\Big\}+O(\epsilon^2)=0. \qquad (45)
\end{aligned}
$$

Let us now equate like powers of $\epsilon^{\frac{1}{2}}$ in (44), (45):

$O(\epsilon^{\frac{1}{2}})$ *terms*. The lowest-order problem for (G_{21}, ϕ_{21}) is easily solved to give

$$G_{21} = Z(\tau)\,e^{iT}+\bar{Z}(\tau)\,e^{-iT}, \qquad (46)$$

$$\phi_{21} = \frac{\Omega}{6}i\{Z\,e^{iT}-\bar{Z}\,e^{-iT}\}, \qquad (47)$$

where $Z(\tau)$ is a complex function to be determined at higher order.

$O(\epsilon)$ *terms*. We define the scalar product of the pair of functions $f(\theta), g(\theta)$ as

$$\int_0^\pi f(\theta)\,g(\theta)\sin\theta d\theta.$$

We then equate terms of order ϵ in (44), (45) and perform the scalar product of the resulting equations with P_0.

The differential problem for (G_{01}, ϕ_{01}) is thus found in the form

$$\phi_{01} = \frac{\tilde{\Omega}}{2}\frac{\partial G_{01}}{\partial T}+\tfrac{6}{5}\phi_{21}G_{21}, \qquad (48)$$

$$\frac{\partial^2 G_{01}}{\partial T^2}+\left(\frac{4\Omega_0^2}{\Omega^2}\right)G_{01} = \frac{4}{\Omega^2}\{-\tfrac{21}{10}(\phi_{21}^2)+(\tfrac{4}{5}\Omega_0^2-3)\,(G_{21})^2\}. \qquad (49)$$

By means of (46), (47) the equations (48) and (49) are readily solved to give

$$G_{01} = a_{01}Z\bar{Z}+b_{01}(Z^2e^{2iT}+\bar{Z}^2e^{-2iT})+\text{hom}, \qquad (50)$$

$$\phi_{01} = \Omega i(b_{01}+\tfrac{1}{5})\,(Z^2e^{2iT}-\bar{Z}^2e^{-2iT})+\text{hom}, \qquad (51)$$

where

$$a_{01} = \frac{\tilde{\Omega}^2-72}{60\Omega_0^2}, \quad b_{01} = \frac{5\tilde{\Omega}^2-24}{40(\Omega_0^2-\tilde{\Omega}^2)}, \qquad (52)$$

and hom denotes the homogeneous solution of (48), (49). This contribution is not relevant to the present analysis which is restricted to values of $\tilde{\Omega}$ which do not fall within any subharmonic or ultraharmonic resonance regions.

The scalar product of P_2 with the $O(\epsilon)$ equations leads to a differential problem for (G_{22}, Φ_{22}), which can be reduced to

$$\phi_{22} = \frac{\tilde{\Omega}}{6}\frac{\partial G_{22}}{\partial T} + \tfrac{6}{7}G_{21}\phi_{21}, \tag{53}$$

$$\frac{\partial^2 G_{22}}{\partial T^2} + G_{22} = -\frac{72}{7\tilde{\Omega}^2}(\phi_{21})^2 + \tfrac{5}{7}(G_{21})^2. \tag{54}$$

Using (46) and (47) the previous system can be solved to obtain

$$G_{22} = \tfrac{6}{7}(Z\bar{Z}) - \tfrac{1}{3}(Z^2 e^{2iT} + \bar{Z}^2 e^{-2iT}) + \text{hom}, \tag{55}$$

$$\phi_{22} = \tfrac{2}{63}\tilde{\Omega}i(Z^2 e^{2iT} - \bar{Z}^2 e^{-2iT}) + \text{hom}, \tag{56}$$

where the homogeneous solution can be ignored again for the present purposes.

Finally the scalar products of the $O(\epsilon)$ equations with P_4 are evaluated. The resulting system for (G_{41}, ϕ_{41}) is

$$\phi_{41} = \frac{\tilde{\Omega}}{10}\frac{\partial G_{41}}{\partial T} + \tfrac{288}{175}G_{21}\phi_{21}, \tag{57}$$

$$\frac{\partial^2 G_{41}}{\partial T^2} + \tfrac{15}{2}G_{41} = -\frac{2556}{35\tilde{\Omega}^2}(\phi_{21})^2 + \tfrac{81}{35}(G_{21})^2, \tag{58}$$

and admits the solution

$$G_{41} = \tfrac{4}{105}(2Z\bar{Z}) + \tfrac{304}{245}(Z^2 e^{2iT} + \bar{Z}^2 e^{-2iT}) + \text{hom},$$
$$\phi_{41} = \tfrac{128}{245}\tilde{\Omega}i(Z^2 e^{2iT} - \bar{Z}^2 e^{-2iT}) + \text{hom}. \tag{59a, b}$$

$O(\epsilon^{\frac{3}{2}})$ terms. We then equate terms of order $\epsilon^{\frac{3}{2}}$ in (44), (45). The pair of equations thus obtained is then multiplied by P_2 and the scalar product evaluated. The resulting equation for G_{23} can be written in the form

$$\begin{aligned}\frac{\partial^2 G_{23}}{\partial T^2} + G_{23} = {} & -\rho_1 G_{21} - \frac{18}{\tilde{\Omega}}\frac{\partial \rho_1'}{\partial T}\phi_{21} + 3G_{01}G_{21} - \frac{24}{\tilde{\Omega}}\frac{\partial G_{01}}{\partial T}\phi_{21} + \tfrac{12}{7}G_{21}G_{22} \\ & - \frac{48}{7.\tilde{\Omega}}\frac{\partial G_{22}}{\partial T}\phi_{21} + \tfrac{26}{7}G_{21}G_{41} - \frac{48}{7.\tilde{\Omega}}\frac{\partial G_{41}}{\partial T}\phi_{21} + \frac{12}{\tilde{\Omega}^2}\phi_{01}\phi_{21} \\ & + \frac{2}{\tilde{\Omega}}G_{21}\frac{\partial \phi_{01}}{\partial T} - \frac{72}{7.\tilde{\Omega}^2}\phi_{21}\phi_{22} + \frac{12}{7.\tilde{\Omega}}\phi_{21}\frac{\partial G_{22}}{\partial T} + \frac{120}{7.\tilde{\Omega}^2}\phi_{21}\phi_{41} \\ & - \frac{20}{7.\tilde{\Omega}}G_{21}\frac{\partial \phi_{41}}{\partial T} + \frac{144}{7.\tilde{\Omega}^2}\phi_{21}\phi_{22} + \frac{12}{7.\tilde{\Omega}}G_{22}\frac{\partial \phi_{21}}{\partial T} + \frac{40}{7.\tilde{\Omega}}G_{41}\frac{\partial \phi_{21}}{\partial T} \\ & - \tfrac{5}{7}(G_{21})^3 - \frac{144}{7.\tilde{\Omega}^2}G_{21}(\phi_{21}')^2 - \frac{12}{\tilde{\Omega}}\frac{\partial \phi_{21}}{\partial \tau} - \frac{12}{\tilde{\Omega}}\lambda\frac{\partial \phi_{21}}{\partial T}.\end{aligned} \tag{60}$$

On substituting from the $O(\epsilon^{\frac{1}{2}})$ and $O(\epsilon)$ solutions into (60) the right-hand side of the resulting equation is found to contain terms proportional to e^{iT} and e^{-iT}. These terms force the 'natural' solutions of (60) thus leading to secular terms unless an orthogonality condition is satisfied. The required condition is

$$2i\frac{dZ}{d\tau} = -\tfrac{7}{2}\tilde{R}_1\bar{Z} + 2\lambda Z + \{5{\cdot}90 + 6a_{01} - 7b_{01}\}Z^2\bar{Z}. \tag{61}$$

The solutions of (61) will be discussed in the next section.

(b) Synchronous case

We now suppose that a and Ω are given by (35) and (36) so that the neutral mode of linear stability theory is synchronous with the basic flow. We again determine how the fundamental mode $n = 2$ arises as a bifurcation from the basic flow. The interaction of this mode with itself and its higher harmonics is identical to that described above in detail for the subharmonic case. However, the fundamental mode must now interact with the basic flow twice before it is reproduced. Thus the scalings used for the subharmonic case must be altered and we therefore take the slow time variable $\tau = \epsilon^2 t$. We then expand F and ϕ in the form

$$\left.\begin{aligned} F(\theta,\phi,T,\tau) &= r-(1+\epsilon f_1+\epsilon^2 f_2-\epsilon^3 f_3,\ldots) = 0,\\ \phi &= R^2\dot{R}r^{-1}+\epsilon\frac{\phi_1}{r}+\epsilon^2\left(\frac{\phi_0}{r}+\frac{\phi_2}{r^3}+\frac{\phi_4}{r^5}\right)+\frac{\epsilon^2\phi_3}{r^3}+\ldots\end{aligned}\right\} \tag{62}$$

where the coefficients in these expansions are as given by (41) and (42).

We can now substitute from (62) into the kinematic and dynamic boundary conditions to give equations similar to (44) and (45). We can then equate terms of order ϵ, ϵ^2, etc. and determine the possible motions of the system. This tedious procedure is similar to that described above for the subharmonic case and so we only give the essential details here.

At order ϵ we find that the first-order function pair (G_{21}, ϕ_{21}) is given by

$$\left.\begin{aligned} G_{21} &= Ze^{2iT}+\bar{Z}e^{-2iT},\\ \phi_{21} &= \frac{\Omega i}{3}(Ze^{2iT}-\bar{Z}e^{-2iT}),\end{aligned}\right\} \tag{63}$$

where $Z(\tau)$ is an amplitude function of the slow time variable τ to be determined at higher order. In fact $Z(\tau)$ is determined as a solvability condition on the order ϵ^3 differential system for (f_3, g_3). This condition gives

$$i\frac{dZ}{d\tau} = \tfrac{1}{4}\left(8\lambda+\frac{b_0^2}{24}-c_0\right)Z+\left(\frac{b_0^2}{16}-\frac{d_0}{2}\right)\bar{Z}-\tfrac{4}{3}R_1(Z^2+2|Z|^2)+(5{\cdot}9+6a_{01}-7b_{01})Z^2\bar{Z} \tag{64}$$

where b_0, c_0, d_0, a_{01} and b_{01} are as defined by (26*d*), (26*b*), (26*c*) with Ω replaced by $\frac{1}{2}\tilde{\Omega}$ and (52) respectively.

We note that, apart from the quadratic terms, the amplitude equation has linear and the usual cubic nonlinear terms. The quadratic terms arise from the interaction of the basic flow and the order ϵ^2 terms in the expansions of F and ϕ.

6. Solution of the amplitude equations

(a) Subharmonic case

The amplitude equation (61) can be solved analytically by defining

$$Z = \alpha e^{-i\theta}. \tag{65}$$

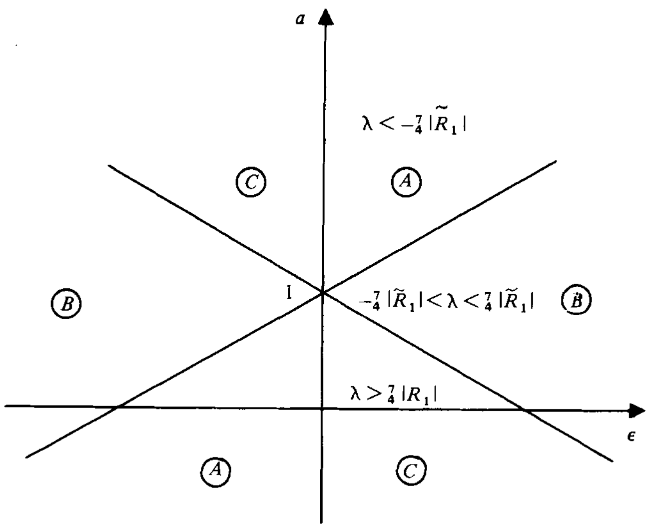

FIGURE 2. Regions of the (a, ϵ) plane relevant to the discussion of the solutions of the amplitude equation for the subharmonic case.

In fact the complex equation (61) is equivalent to the following system of two real equations for α and θ

$$\frac{d\alpha^2}{d\tau} = \tfrac{7}{2}\tilde{R}_1\alpha^2 \sin 2\theta, \tag{66}$$

$$\frac{d(\alpha^2 \cos 2\theta)}{d\tau} = (2\lambda + \alpha^2 e)\,\alpha^2 \sin 2\theta, \tag{67}$$

where we have denoted by e the coefficient of the cubic term of (61).

The solution of the system (66), (67) corresponding to initial conditions (α_0, θ_0) is given in implicit form by

$$\frac{\beta(\theta) \pm (\beta^2(\theta) + C)^{\frac{1}{2}}}{\beta(\theta_0) \pm (\beta^2(\theta_0) + C)^{\frac{1}{2}}} = \exp\left(\tfrac{7}{2}\tilde{R}_1 \sin 2\theta\tau\right), \tag{68}$$

$$e\alpha^2 = \beta(\theta) \pm (\beta^2(\theta) + C)^{\frac{1}{2}} \tag{69}$$

with $\mu(\theta)$ and C respectively defined as

$$\beta(\theta) = -2\lambda + \tfrac{7}{2}\tilde{R}_1 \cos 2\theta, \tag{70}$$

$$C = e\alpha_0^2(e\alpha_0^2 + 4\lambda - 7\tilde{R}_1 \cos 2\theta_0). \tag{71}$$

The solution (68), (69) is bounded and periodic. Its nature depends on the sign of e and on the values of $(\lambda, \tilde{R}_1)$. This can more clearly be appreciated if we examine the nature of the singular points of (61). We refer to the three regions of the (a, ϵ) plane denoted by A, B, C in figure 2.

The origin is a singular point of (61). It is easily shown to be a centre in the subcritical regions A, C and a saddle point in the supercritical regions B. This result is obvious and merely confirms that the basic flow is stable to infinitesimal disturbances in A, C, unstable in B as already known from the linear analysis. We are interested in investigating the existence of further stable finite amplitude solutions. They correspond to further singular points of (61). There are four possible singular points defined by

$$\alpha^2 = \begin{cases}(-2\lambda+\tfrac{7}{2}\tilde{R}_1)/e & (\theta = 0, \theta = \pi),\\ (-2\lambda-\tfrac{7}{2}\tilde{R}_1)/e & (\theta = \tfrac{1}{2}\pi, \theta = \tfrac{3}{2}\pi).\end{cases} \tag{72}$$

For a given sign of e their existence clearly depends on which region of the (a, ϵ) plane we are considering.

If we assume that e is positive we can see that there are two more stable finite-amplitude solutions bifurcating subcritically in the regions A and supercritically in the regions B. These solutions lie either on the real or on the imaginary axis depending on whether the bubble is excited by the acoustic wave above or below volume resonance. Thus above resonance the bubble responds with finite-amplitude inviscid motion which is dominantly (i.e. at leading order) in phase with the excitation, whereas below resonance a phase difference of $\frac{1}{2}\pi$ exists.

The stable finite-amplitude solutions of regions A and B correspond to centres and are located on the real (imaginary) axis for $\tilde{R}_1 > (<) 0$. Two further singular points (saddle points) are present in region A and are located on the imaginary (real axis for $\tilde{R}_1 > (<) 0$.

If, for a given value of λ less than $(-\frac{7}{4}|\tilde{R}_1|)$, we decrease ϵ starting from a small positive value, the amplitude of the two stable subcritical solutions decrease till they merge for ϵ equal to 0 into the origin which remains stable; further decrease of ϵ leads to a pattern identical to that obtained for positive ϵ. Alternatively if we fix a value of ϵ, say positive, and increase λ starting from a negative value less than $(-\frac{7}{4}|\tilde{R}_1|)$ the amplitudes of the subcritical solutions decrease and the zero solution is stable; when the value $(-\frac{7}{4}|R_1|)$ is reached the saddle points merge into the origin which becomes unstable and the amplitudes of the two finite solutions continues to decrease as λ increases till they merge into the origin (when λ equals $(\frac{7}{4}|R_1|)$), which becomes stable.

The sign of e also affects the solution. In fact for negative values of e one can readily show that two stable finite-amplitude solutions exist in regions C, B, and they lie on the imaginary or real axis depending on whether the exciting frequency is above or below volume resonance. Thus the role of the sign of e is merely to reverse the response of the bubble to the acoustic excitation.

Figure 4 shows that e is positive for a large range of values of Ω_0. Thus, in the present problem, no finite amplitude solution is found to occur in regions C, whereas two finite amplitude solutions occur in regions A and B.

(*b*) *Synchronous case*

We now investigate the behaviour of the solution of (64) when λ is varied. It is convenient at this stage to define $W = -\frac{4}{3}ZR_1$ and write the equation which determines W in the form

$$i\frac{dW}{d\tau} = \gamma W + g\overline{W} + W^2 + 2|W|^2 + hW^2\overline{W}. \tag{73}$$

Here the quantities γ, g and h are defined by

$$\gamma = \frac{\lambda}{2}+\frac{b_0^2}{96}-\frac{c_0}{4}, \quad g = \frac{b_0^2}{16}-\frac{d_0}{2}, \quad h = 9\{5{\cdot}9+6a_{01}-7b_{01}\}/16R_1^2, \tag{74a, b, c}$$

so that we are now interested in the solutions of (73) when the parameter γ varies. The corresponding change in the driving frequency of the pressure oscillation at infinity can be evaluated using (36) and (74). The coefficients g and h appearing in (73)

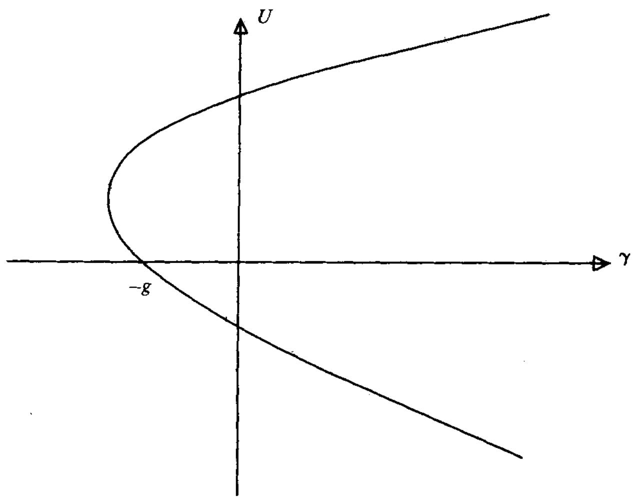

FIGURE 3. The real solutions of (73), the amplitude equation for the synchronous mode for different values of the driving frequency.

take on different values when the parameter Ω_0 varies. For this reason we discuss the solutions of (73) for general values of g and h and we shall later show which solutions are relevant to the physical problem under investigation.

We first consider the equilibrium solutions of (73) and their stability properties. It is easily seen from this equation that, apart from the trivial solution $W = 0$, there are purely real equilibrium solutions given by $W = U$ where U satisfies

$$0 = (\gamma+g)+3U+hU^2. \tag{75}$$

This solution corresponds to the periodic solution $ce_2(t)$ of Mathieu's equation and exists only for $\gamma > -g+9/4h$ or $\gamma < -g+9/4h$ depending on whether $h < 0$ or $h > 0$ respectively. For definiteness we now take h to be negative and the real solutions of (75) are then as shown in figure 3. The stability properties of these solutions depend on the quantities h and g and are closely related to the question of whether or not further solutions of (73) exist. We might expect that such solutions would be purely imaginary corresponding to the periodic solution $se_2(t)$ of the Mathieu equation. However the quadratic terms in (73) prevent this occurrence so any further solutions must be neither purely real nor purely imaginary. Suppose that such equilibrium solutions exist and are of the form $W = U+iV$ with $U \neq V \neq 0$. If we replace W by $U+iV$ in (73) and take real and imaginary parts of this equation we obtain a pair of coupled equations for U and V. Some straight-forward manipulations with these equations shows that the modulus of the complex solutions is determined by

$$|W|^2 = \frac{(\gamma-g)g}{1-hg} \tag{76}$$

so that these solutions only exist for $(\gamma-g)g/(1-hg) > 0$. The real and imaginary parts of this solution are then given by

$$U = -|W|^2/(2g), \tag{77}$$

$$V = \pm\left(1-\frac{|W|^2}{4g^2}\right)^{\frac{1}{2}}|W|. \tag{78}$$

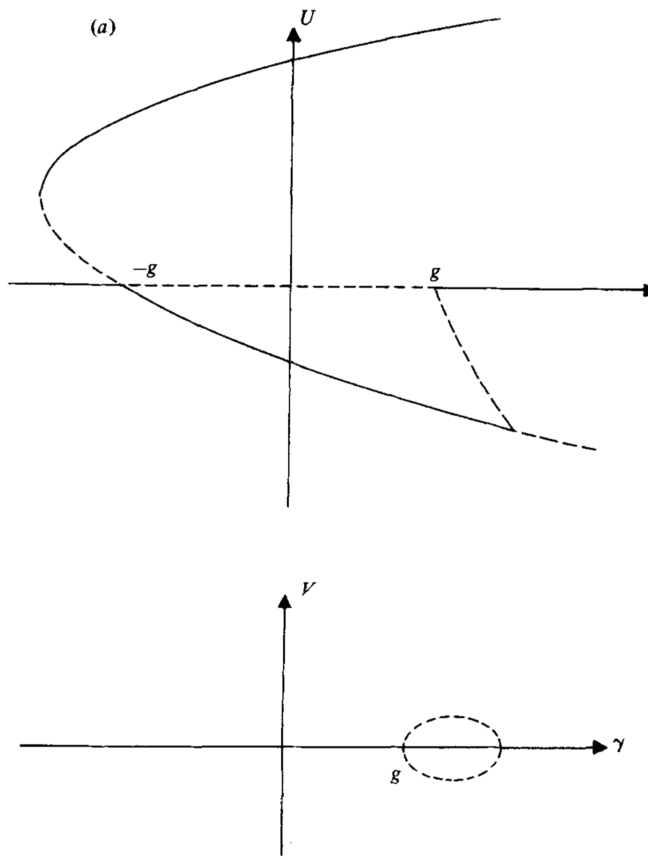

FIGURE 4 (a). For legend see page 17.

Thus a further constraint on γ arises from the condition that $(1 - |W|^2/4g^2)$ is positive. The two conditions effectively restrict the existence of the mixed mode (complex) solution to a finite range of values of λ. This range of values of λ depends on g and h. We again assume that h is negative and it is then necessary to consider the following cases:

$$(a)\ h < 0,\quad g > 0; \quad (b)\ h < 0,\quad 0 > g > \frac{1}{h}; \quad (c)\ h < 0,\quad g < \frac{1}{h} < 0.$$

The nature of the complex solutions and the real solutions for the above cases is shown in figures 4(*a*, *b*, *c*). We have also indicated the stability properties of the different solutions. We note that the complex solutions occur as secondary bifurcations from the real ones. Moreover, since V/U is a function of γ, the phase difference between the mixed mode solution and the basic flow changes with the driving frequency. This is the essential physical difference between the solutions corresponding to real and complex amplitudes. We note that only in the case $g < 1/h < 0$ is the mixed-mode solution stable and even then the range of frequencies for which the solution exists is finite. We postpone a more detailed discussion of figure 4 until we have discussed the solutions of (75) for h positive.

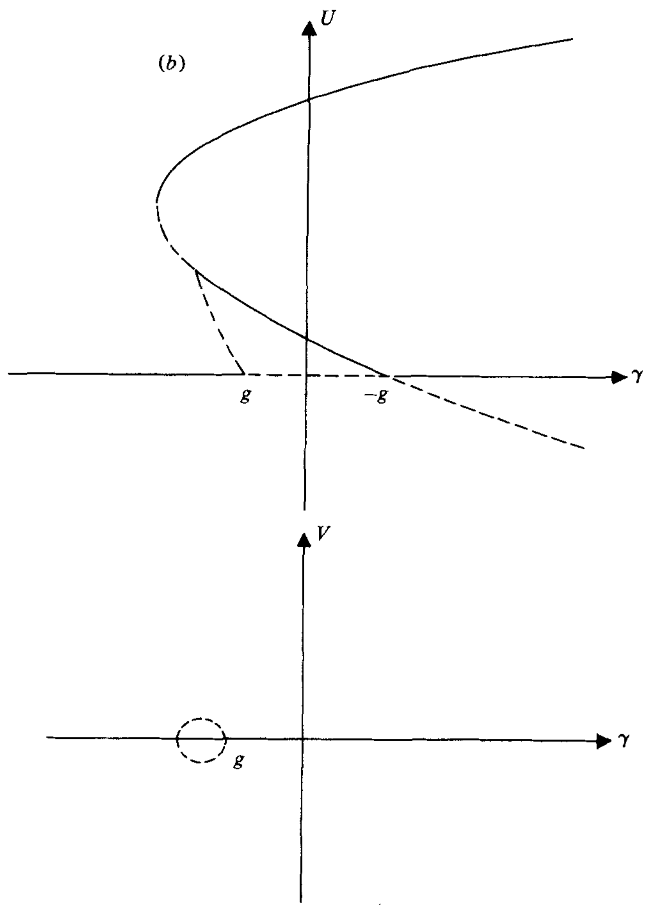

FIGURE 4 (b). For legend see facing page.

If h is now taken to be positive we write $w = -W$ in which case (73) becomes

$$-i\frac{dW}{d\tau} = -\gamma W - gW + W^2 + 2|W|^2 - hW^2\bar{W}. \tag{79}$$

The form of the right-hand side of (79) is identical to that of (73) apart from the sign of the linear and the cubic terms. Thus the previous discussion for $h < 0$ can be applied directly to the case $h > 0$ by changing the signs of γ, g and h in that discussion. We note that the sign of the time derivative on the left-hand sides of (73) and (79) are different but this only changes the direction of the phase plane contours and does not alter any of the stability properties shown in figure 4.

Thus for h positive it is necessary to consider the cases

$$(a)\ h > 0,\quad g < 0;\quad (b)\ 0 < g < \frac{1}{h};\quad (c)\ 0 < \frac{1}{h} < g.$$

The equilibrium solutions of (79) corresponding to these three cases are easily described by reference to figures 4(a, b, c) respectively. This is done simply by replacing U, V, g, h and γ by $-U$, $-V$, $-g$, $-h$, and $-\gamma$ in these figures and their

(c)

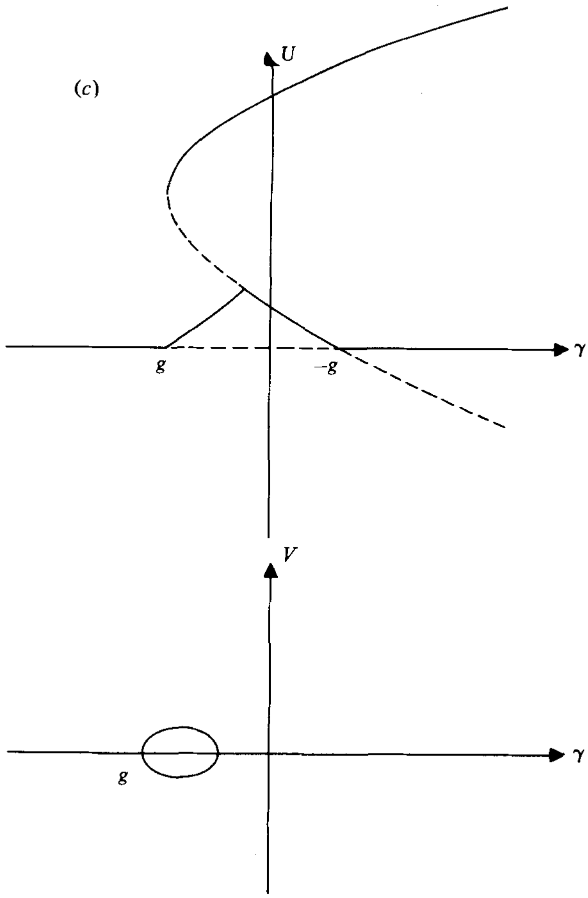

FIGURE 4. The solutions of (73) and their stability properties for different values of the driving frequency. (a) $h < 0$, $g > 0$; (b) $0 < 1/h < 0$; (c) $g < 1/h < 0$. ——, stable solutions; – – –, unstable solutions.

legends. Hence for $h > 0$ increasing the driving frequency corresponds to moving along the horizontal axis from right to left.

It remains now for us to discuss the case $h = 0$ in which case (73) becomes

$$i\frac{dW}{d\tau} = \gamma W + g\overline{W} + W^2 + 2|W|^2.$$

The real equilibrium solutions of the above equation are given by

$$U = -\tfrac{1}{3}(\gamma + g),$$

whilst the mixed mode solutions are given by (76), (77) and (79) with $h = 0$. We again find that the latter solution exists for only a finite range of values for γ. In figures 5(a, b) we have shown the real and complex solutions for $h = 0$. The degenerate case $g = h = 0$ requires a complete rescaling of the finite amplitude solutions but this is not in fact necessary here because, as will be seen shortly, such a situation is not physically relevant. We see that figure 5 can be recognized as limiting forms of

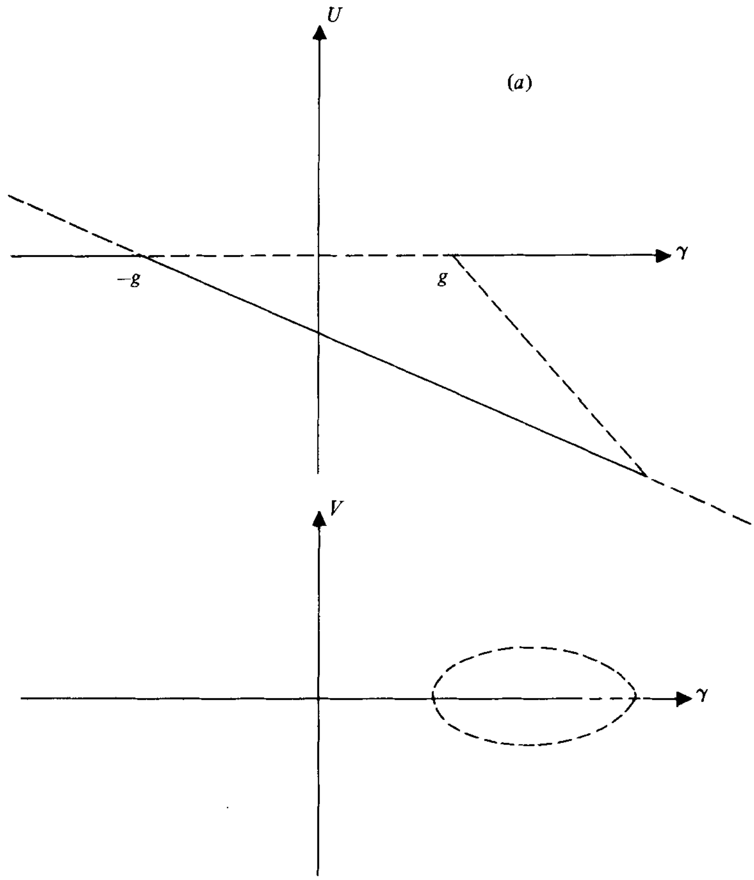

FIGURE 5 (a). For legend see facing page.

figures 4(a, b) respectively with the minimum points of the latter figures having moved off to infinity. We further note that the only non-zero stable solutions of (73) for $h = 0$ are real and that such solutions exist only for a finite range of values for γ.

We are now in a position to discuss in detail the response of the bubble in the synchronous case when Ω_0^2, and hence g, h vary. In figure 6 we have shown g and h as functions of Ω_0^2 for the range $\Omega_0^2 \geqslant 3$. (Note that $\Omega_0^2 < 3$ has no physical relevance.) We see that at the resonance positions, $\Omega_0^2 = 3{\cdot}27, 4{\cdot}5$, either $|g|$ or $|h|$ or both become infinite. The present analysis is therefore not valid in a neighbourhood of these points. We also note that h changes sign at $\Omega_0^2 = 3{\cdot}29814$. For $\Omega_0^2 = 3{\cdot}29814$ the coefficient g is positive so the bubble motion is described by figure 5(a). Hence when the frequency is increased through the unstable range $-g \leqslant \gamma \leqslant g$ the bubble responds with a finite amplitude motion in phase with the driving pressure oscillation. In fact this motion remains stable for $\gamma \leqslant 5g$ and then the finite amplitude motion will change discontinuously since the only stable solution is now $V = 0$. If the frequency is decreased from any value bigger than that corresponding to $\gamma = g$ the zero solution remains stable until $\gamma = g$ and then the motion changes discontinuously to a stable finite amplitude motion in phase with the driving pressure gradient. Thus there is a hysteresis phenomenon associated with changing the driving pressure oscillation frequency.

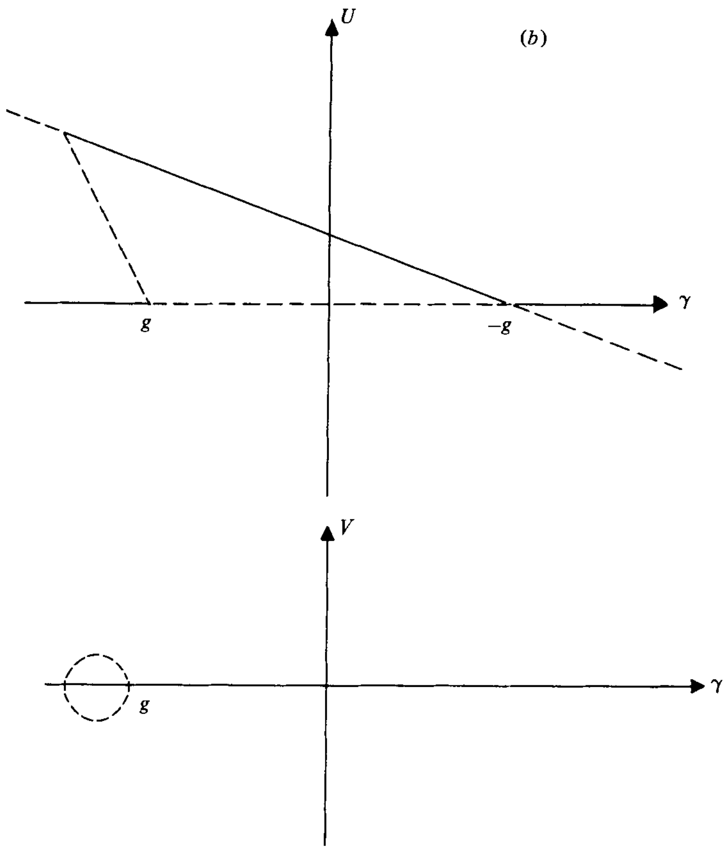

FIGURE 5. The solutions of (73) and their stability properties for different values of the driving frequency. (a) $h = 0, g > 0$; (b) $h = 0, g < 0$. ——, stable solutions; – – –, unstable solutions.

We now discuss the response of the bubble for $\Omega_0^2 > 3{\cdot}29814$ in which case h is always positive. In the interval $(3{\cdot}29814, 3{\cdot}29819)$ the coefficients h and g are such that $0 < g < 1/h$ whilst for $\Omega_0^2 > 3{\cdot}29819$ we have $0 < 1/h < g$. The equilibrium solutions for the former and latter situations are given by figures 4(b, c) respectively with U, V, g, h and γ replaced by $-U$, $-V$, $-g$ and $-\gamma$. In view of the restricted range in which figure 4(b) describes the equilibrium solutions of (73) we can reasonably assume that only the range $\Omega_0^2 > 3{\cdot}29819$ is physically relevant. We can see that in this case the finite amplitude of the bubble motion changes continuously with the frequency. There is no hysteresis phenomenon in this case. Moreover the response of the bubble is sometimes in phase with driving pressure gradient ($U \neq 0, V = 0$) for a certain frequency range and out of phase (U; $V \neq 0$) otherwise.

Thus to summarize we can say that, apart from the resonance regions, the bubble response to a change in the driving frequency depends on whether $\Omega_0^2 > 3{\cdot}29819$ or $\Omega_0^2 < 3{\cdot}29814$. In the former case no hysteresis is to be expected but phase differences between the driving pressure gradient and the nonspherical oscillations of the bubble will occur for certain frequencies. In the latter case no such phase difference is predicted but hystereses is to be expected.

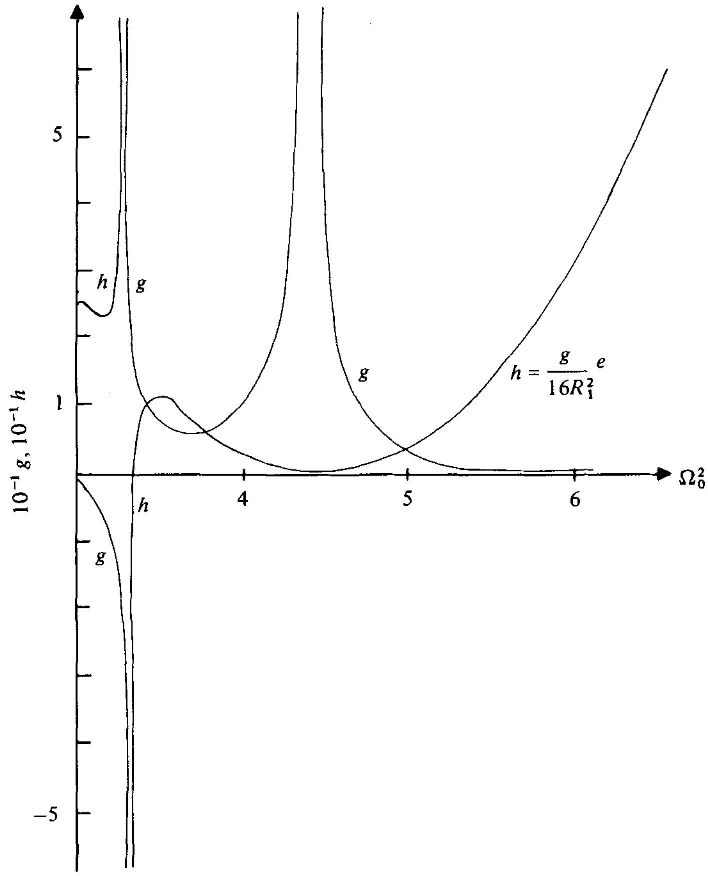

FIGURE 6. The coefficients h, g of the amplitude equation of the synchronous mode plotted versus Ω_0^2.

7. Discussion

The analysis of this paper shows how small amplitude surface oscillations of a bubble are induced by an acoustic field. In particular we have investigated in detail the cases when the response of the bubble is subharmonic or synchronous with the basic flow. We have constructed asymptotic representations of the finite amplitude motion in terms of the parameter ϵ which essentially represents the difference between the driving frequency and the frequency at which linear theory predicts the subharmonic (or synchronous) mode should become unstable.

There are important differences between the two types of response. The amplitude of the steady bifurcating solutions is larger ($O(\epsilon^{\frac{1}{2}})$) in the subharmonic case than in the synchronous case $O((\epsilon))$. Furthermore a subharmonic response is found to lead to a symmetric bifurcation whereas transcritical bifurcation occurs for a synchronous response. The latter case also suffers secondary bifurcation for some values of the relevant parameters of the problem.

The above differences are obviously related to the different structures of the amplitude equations in the two cases. Indeed quadratic terms are only present in (61) and they cause the occurrence of transcritical and secondary bifurcation. We notice that such quadratic terms originate from the interaction between the basic flow (at lowest order) and the first-order perturbations.

The synchronous case exhibits a further interesting feature. Unlike the subharmonic case, a steady streaming component is present in the second-order perturbations. Such a component comes from the interaction between the first-order basic flow and the leading order perturbation. Whether or not this steady streaming effect is related to Elder's (1959) and Gould's (1966) findings cannot be stated. In fact both of these experimental works refer to bubbles attached to walls, and show various streaming regimes depending on the sonic amplitude. At least in one of these regimes the streaming appears to be associated with the occurrence of non spherical surface oscillations. However the presence of the wall cannot be ignored and its influence does not seem to be easily assessed.

The onset of an erratic motion of the bubble is not explicable under the conditions examined in the present work. However one can readily see that the nonlinear interactions following the linear growth of the spherical harmonic P_3 would lead at second order to the excitation of the harmonic P_1, which merely consists of a displacement of the centre of gravity of the bubble. Alternatively a more accurate representation of the sound field would lead to a θ-dependent basic flow which would interact with the perturbations proportional to P_2 to produce P_1. We do not pursue either of these possibilities in more detail here.

The finite amplitude solutions discussed in the previous section have been found to be periodic both in the fast time scale imposed by the acoustic excitation and in the slow time scale associated with the linear growth. This results from neglecting any possible source of damping (viscous, acoustic, thermal) present in the actual phenomenon. However one can safely take the finite amplitude steady state solutions derived in the previous sections as good estimates of the actual equilibrium solutions provided:

(i) the size of the cavity is much smaller than the acoustic wavelengths both in the gas and in the liquid;

(ii) vorticity is confined within a layer of characteristic thickness $(2\nu/\omega)^{\frac{1}{2}}$, much smaller than the average radius of the bubble.

We notice that the assumption of rotational symmetry of the perturbations has considerably reduced the amount of algebraic work involved. However the structure of the nonlinear expansions (45), (46) and (69) is not crucially altered if the leading order perturbation contains ϕ-dependent terms proportional to P_2^m $(m = 0, \pm 1, \pm 2)$. In this case the analysis would lead to a set of five complex nonlinear equations for the amplitudes of the fundamental. The derivation of the coefficients of this system requires an enormous amount of algebra.

Finally we recall that the present analysis is restricted to values of Ω which are not close to an integral fraction or a multiple of Ω_0.

In such resonant regions the basic flow is no longer given in (15)–(18) (see Prosperetti 1974) and may contain an $O(1)$ forcing.

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